Site percolation on planar graphs and circle packings

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on sabbatical at the IAS and Princeton University

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Site percolation

- Site percolation with probability $0 < p < 1$ on a (simple, connected) graph G is the random subgraph G_p formed by independently retaining each vertex of G with probability p, and otherwise deleting it.

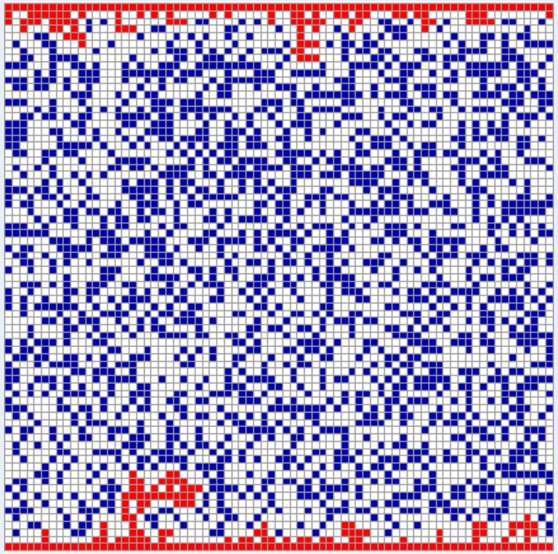
- Research focuses on the connected components (clusters) of G_p.
  The probability that G_p contains an infinite cluster transitions from zero to one at a critical value $p_c \in [0,1]$ (the probability is 0 for $p < p_c$ and is 1 for $p > p_c$).

- Percolation is classically studied on structured graphs including lattices such as $\mathbb{Z}^d$, the complete graph (Erdős–Rényi $G(n,p)$) or regular trees, but the case of general graphs is also of great interest.

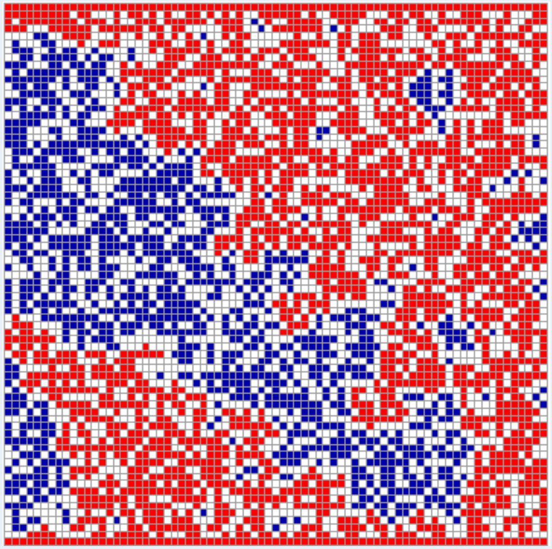
- In this talk we discuss percolation on infinite planar graphs and focus on the value of p_c.

Site percolation on $\mathbb{Z}^2$

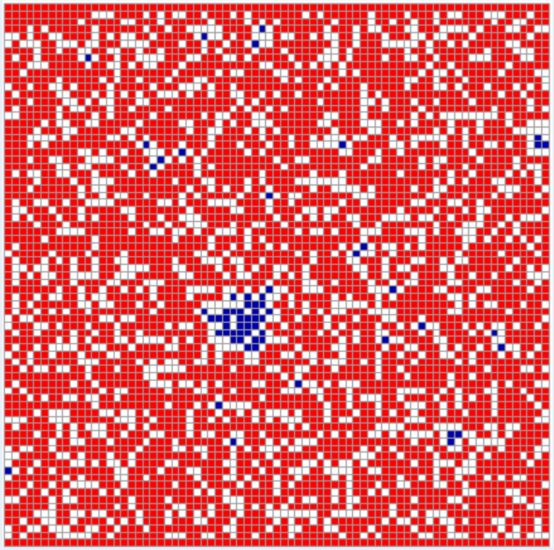
- Simulations of site percolation on a 75×75 grid in the square lattice $\mathbb{Z}^2$ (from Wolfram demonstrations project).
- Clusters of bottom and top highlighted in red.
- Site percolation threshold $p_c \approx 0.59274(10)$ (Derrida–Stauffer 1985).



p=0.4

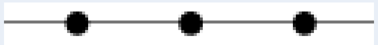


p=0.59

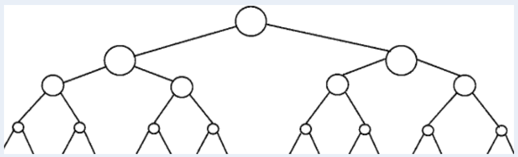


p=0.7

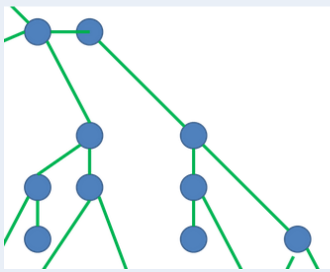
Infinite planar graphs



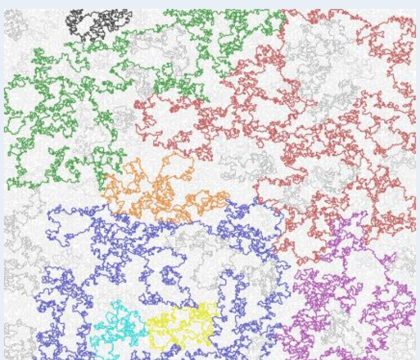
$\mathbb{Z}, p_c = 1$



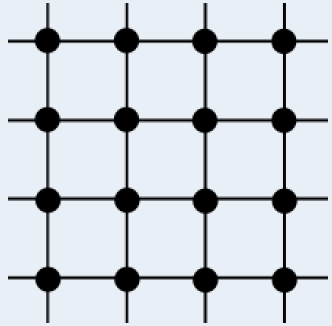
Binary tree, $p_c = 1/2$



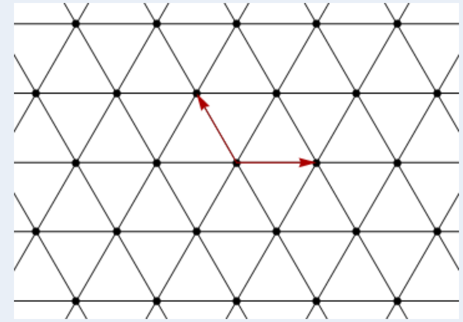
General tree



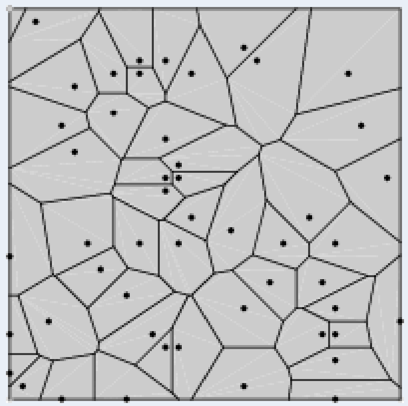
The random loops in the loop O(n) model
Simulation by Y. Spinka



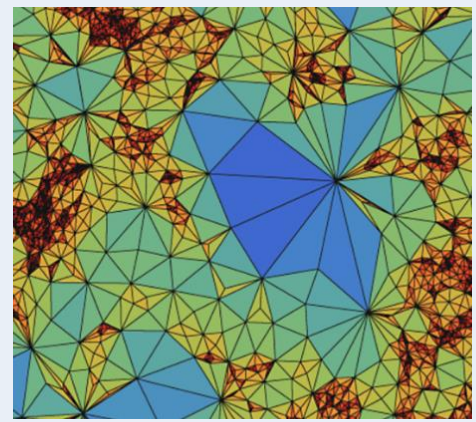
$\mathbb{Z}^2, p_c \approx 0.59$



$\mathbb{T}, p_c = 1/2$



Voronoi diagram
of points in $\mathbb{R}^2$



Random triangulations
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Benjamini conjectures

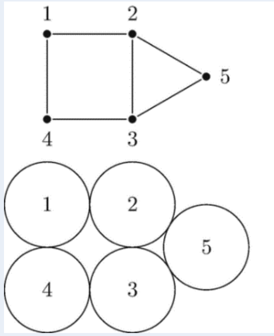
- How high or low can the value of p_c be for planar graphs?
- General lower bound: $p_c(G) \geq \frac{1}{\Delta(G)-1}$ where $\Delta(G)$ is the maximal degree of G.
  Follows from union bound: At most $\Delta(G)(\Delta(G)-1)^{L-2}$ paths of length L from each vertex. Sharp for regular trees, so p_c can be arbitrarily low for planar graphs. There also exist planar graphs with $p_c < 1$ but arbitrarily close to 1 (e.g., sub-divide edges of $\mathbb{Z}^2$).
- Can we say more for special classes of planar graphs?
  Benjamini (2018) relates the question to the behavior of simple random walk on G.
- G is *recurrent* if simple random walk on it returns to its starting point infinitely often. Otherwise, G is *transient*.
  G is *one-ended* if it has a unique infinite connected component after removing any finite set of vertices (e.g., $\mathbb{Z}^2$ is one-ended but $\mathbb{Z}$ is not).
  A planar graph G is a *triangulation* if it has a planar drawing with all faces triangles.
- For site percolation with $p = 1/2$ on bounded-degree one-ended triangulations G:
  Conjecture (Benjamini): If G is transient then $G_{1/2}$ has an infinite cluster.
  Question (Benjamini): Does recurrence of G imply that $G_{1/2}$ has no infinite cluster?
  In particular, p_c cannot be arbitrarily high/low for such planar graphs.

Results

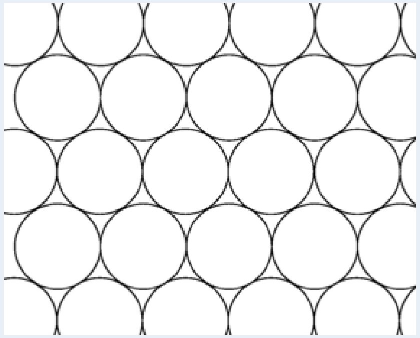
- Theorem (P. 2020): There exists $p_0 > 0$ such that the following holds for all one-ended triangulations G:
  - If G is recurrent then G_{p_0} has no infinite cluster.
  - If G is bounded degree and transient then G_{1-p_0} has an infinite cluster.
- The emphasis is that p_0 is universal - p_c is uniformly bounded on such graphs.
- Verifies Benjamini's predictions when the probability $p = 1/2$ is replaced by a sufficiently low/high (but fixed!) probability.
- Recurrent case does not require G to be of bounded degree.

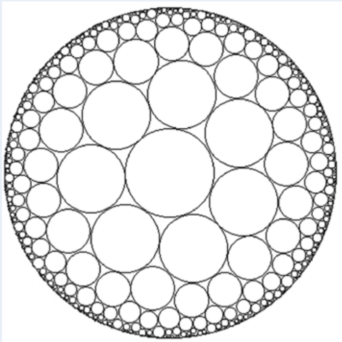
Tool: Circle Packings

- A circle packing P is a collection of closed disks in $\mathbb{R}^2$ with disjoint interiors.
  Graph structure on P: disks adjacent if tangent.
  Accumulation points of disks are allowed.
  After site percolation with probability p, write P_p for the graph of retained disks.
  Carrier of a circle packing representing a triangulation: Union of disks and interstices between disks.
- Theorem (Koebe 1936, Andreev, Thurston): Every (finite or infinite, simple) planar graph can be represented by a circle packing.



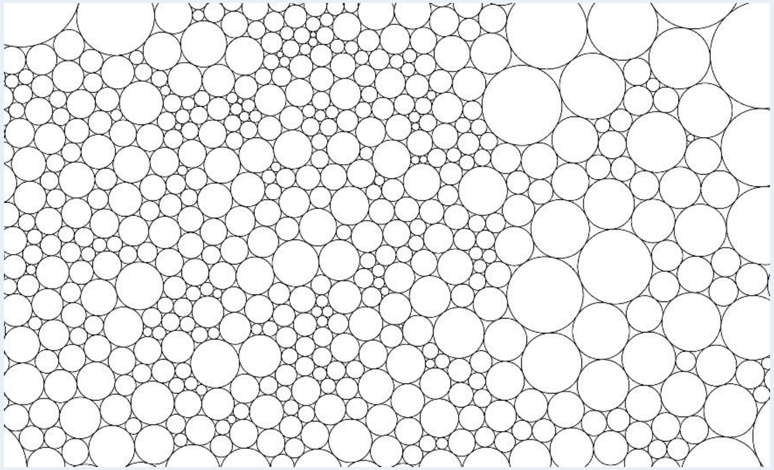
Circle packing representing a graph


Carrier = $\mathbb{R}^2$

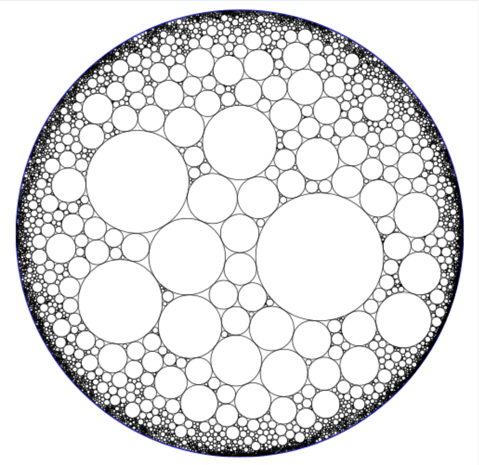

Carrier = unit disk $\mathbb{D}$

Circle packings and recurrence

- Theorem (He–Schramm 1995, uniformization theorem):
  Let G be a one-ended triangulation. Then
  1) G may be represented by a circle packing with carrier $\mathbb{R}^2$ (CP-parabolic) or by a circle packing with carrier $\mathbb{D}$ (CP-hyperbolic), but not by both.
  2) If G is recurrent then it is CP-parabolic.
  3) If G is of bounded degree and transient then it is CP-hyperbolic.



Carrier = $\mathbb{R}^2$
CP-parabolic

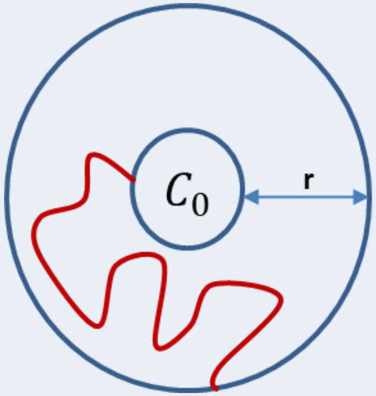


Carrier = unit disk $\mathbb{D}$
CP-hyperbolic

Main result

- Theorem (P. 2020): There exists $p_0 > 0$ such that for every circle packing P:
  - The retained graph P_{p_0} contains no cluster of infinite (Euclidean) diameter.
  - Moreover, if $D \coloneqq \sup_{C \in P} \mathrm{diam}(C) < \infty$ then for each disk $C_0 \in P$,

$$\mathbb{P}(C_0 \text{ is connected to distance } r \text{ after percolation}) \le e^{-\frac{r}{D}}$$



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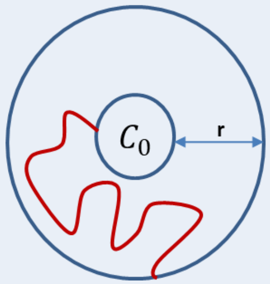
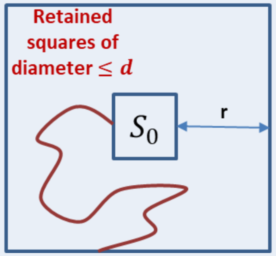


- Theorem says that while infinite clusters may exist after p_0 site percolation, they necessarily connect to accumulation points rather than to infinity.
- Result on recurrent one-ended triangulations follows as they are represented by circle packings with no accumulation points by the He-Schramm theorem. Transient case follows from the He-Schramm theorem and the quantitative bound above using an additional argument.
- Conjecture that p_0 may be taken to be 1/2 in the first part of the theorem. Will imply a positive answer to Benjamini's question on recurrent triangulations.

Proof 1 (statement to prove)

- Technically convenient to prove result for square packings (circle packing case requires minor alterations). Convenient to draw pictures with ℓ_∞-distance.
  Square packing: a collection P of closed squares in $\mathbb{R}^2$ with disjoint interiors.
  Graph on P: squares adjacent if they intersect.
  Percolation: Retain each square with a small probability p independently ($p = e^{-26}$ is sufficiently small for the argument).
- Write $\{S_0 \overset{\leq d}{\rightarrow} r\}$ for the event that the square S_0 is connected to distance r by retained squares whose diameters do not exceed d.



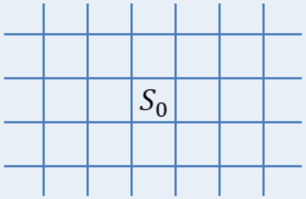
- Prove following result (general case is similar):
  Let P be a square packing with squares of diameter at least 1.
  For each $r > 0$, integer $k \geq 0$ and $S_0 \in P$ with $\mathrm{diam}(S_0) \leq 2^k$ have

$$\mathbb{P}\left(S_0 \overset{\leq 2^k}{\longrightarrow} r\right) \leq e^{-\frac{r}{2^k}}.$$

Proof 2 (induction base)

- To prove: Let P be a square packing with squares of diameter at least 1.
  For each $r > 0$, integer $k \geq 0$ and $S_0 \in P$ with $\mathrm{diam}(S_0) \leq 2^k$ have
  $$\mathbb{P}\left(S_0 \overset{\leq 2^k}{\longrightarrow} r\right) \leq e^{-\frac{r}{2^k}}.$$

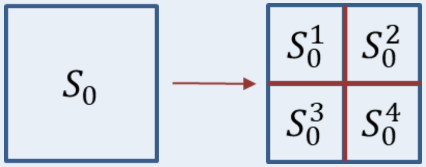
- Proof by double induction on k and r.
- Base case $k = 0$: at most 8 neighbors to each square.
  Expected number of paths in P_p of length L is at most $8^{L-1} \cdot p^L$.
  Need path of length $\lceil r \rceil$ to reach distance r.
  Probability at most e^{-r} when $p \leq \frac{1}{8} e^{-1}$.



- Induction hypothesis 1: Fix $k \geq 1$ and assume result known for $k - 1$ and all r.
- Base case $r \leq 2^{k+1}$: $\mathbb{P}\left(S_0 \overset{\leq 2^k}{\longrightarrow} r\right) \leq \mathbb{P}(S_0 \text{ is retained}) = p \leq e^{-2} \leq e^{-\frac{r}{2^k}}$.
- Induction hypothesis 2: Fix $r > 2^{k+1}$ and assume result up to $r - 2^k$.

Proof 3 (diameter of S_0)

- Diameter of S_0: To use induction, wish to ensure that $\mathrm{diam}(S_0) \leq 2^{k-1}$. If this is not already the case:
  - Cut S_0 into four squares $S_0^1, S_0^2, S_0^3, S_0^4$.



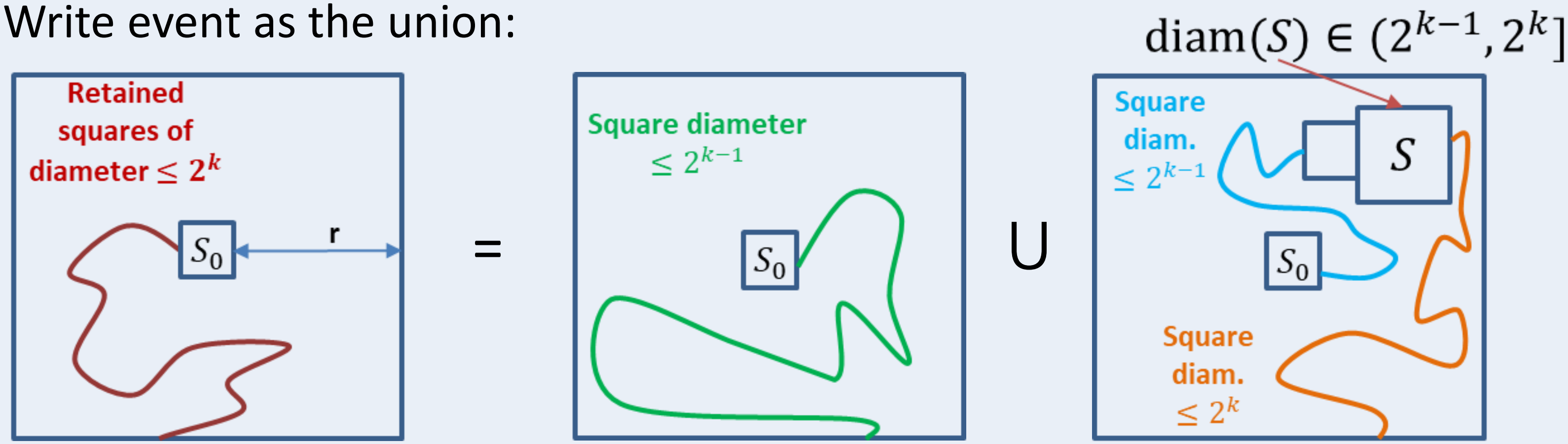
  - Replace (P, S_0) by (P^i, S_0^i), for $1 \leq i \leq 4$, where $P^i = (P \setminus \{S_0\}) \cup \{S_0^i\}$.
  - Prove the slightly stronger bound

$$\mathbb{P}\left(S_0^i \xrightarrow{\leq 2^k} r\right) \leq \frac{1}{4} e^{-\frac{r}{2^k}}.$$

- This will establish the result for (P, S_0) by a simple coupling with the percolations on the $\left(P^i, S_0^i\right)$.

Proof 4 (using the induction)

- Recap: P a square packing with squares of diameter at least 1. Fix $k \geq 1$.
  To prove: For each $r > 0$ and $S_0 \in P$ with $\text{diam}(S_0) \leq 2^{k-1}$
  $$\mathbb{P}\left(S_0 \xrightarrow{\leq 2^k} r\right) \leq \frac{1}{4} e^{-\frac{r}{2^k}}.$$
  Bound holds by induction (without ¼) for $k-1$ and all r.
  Fix $r > 2^{k+1}$. Bound holds by induction (without ¼) up to $r - 2^k$ (for fixed k).
- Write event as the union:

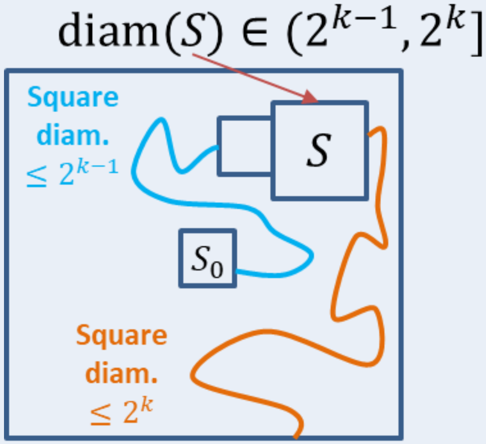


$\left\{S_0 \xrightarrow{\leq 2^k} r\right\}$ $\quad$ $\left\{S_0 \xrightarrow{\leq 2^{k-1}} r\right\}$ $\quad$ On next slide

- Note $\mathbb{P}\left(S_0 \xrightarrow{\leq 2^{k-1}} r\right) \leq e^{-\frac{r}{2^{k-1}}} \leq e^{-2} \cdot e^{-\frac{r}{2^k}}$ using the induction and $r > 2^{k+1}$.

Proof 5 (using the induction 2)

- Second event states that there exists $S \in P$ with $d(S_0, S) \leq r$ and $\mathrm{diam}(S) \in (2^{k-1}, 2^k]$ such that
  - S_0 connects to a neighbor of S with retained squares of diameter at most 2^{k-1},
  - S connects to distance r from S_0 with retained squares of diameter at most 2^k,
  - These connections use a disjoint set of squares.



- By BK inequality and induction hypotheses, this event has probability at most

$$\mathbb{P}\left(S_0 \xrightarrow{\leq 2^{k-1}} d(S_0,S) - 2^{k-1}\right) \cdot \mathbb{P}\left(S \xrightarrow{\leq 2^k} r - d(S_0,S) - 2^k\right)$$

$$\leq \min\left\{\exp\left(-\frac{d(S_0,S) - 2^{k-1}}{2^{k-1}}\right), p\right\} \cdot \exp\left(-\frac{r - d(S_0,S) - 2^k}{2^k}\right)$$

$$\leq \min\left\{\mathrm{e}^2 \cdot \mathrm{e}^{-\frac{d(S_0,S)}{2^k}}, p \cdot e^{1+\frac{d(S_0,S)}{2^k}}\right\} \cdot e^{-\frac{r}{2^k}}$$

Proof 6 (finish)

- We have obtained

$$\mathbb{P}\left(S_0 \xrightarrow{\leq 2^k} r\right) \leq \left(e^{-2} + \sum_{S} \min\left\{\mathrm{e}^2 \cdot \mathrm{e}^{-\frac{d(S_0,S)}{2^k}}, p \cdot e^{1+\frac{d(S_0,S)}{2^k}}\right\}\right) e^{-\frac{r}{2^k}}$$

  where the sum is over all $S \in P$ with $d(S_0, S) \leq r$ and $\mathrm{diam}(S) \in \left(2^{k-1}, 2^k\right]$.

- It remains to note that by area considerations, the number of such S with $d(S_0, S) \leq m \cdot 2^k$ is of order at most m^2.

- It follows that the expression in the parenthesis is at most $\frac{1}{4}$ when p is sufficiently small, finishing the proof by induction.

- Remark: The area considerations are the only place in the argument where the fact that we have squares rather than, say, rectangles, is used.

Extensions

- Theorem (P. 2021+): Let (X, d) be a metric space and P be a countable collection of subsets of X of finite diameter (not necessarily a packing). Graph structure on P: Sets adjacent if have non-empty intersection. Suppose that for each $S \in P$ and $\rho, t > 0$,

$$|\{S' \in P : d(S, S') \leq \rho, \mathrm{diam}(S') \geq t\}| \leq e^{C_1 + C_2 \frac{\rho + \mathrm{diam}(S)}{t}}$$

  for some $C_1, C_2 > 0$. Then there exists $p > 0$ depending only on C_1, C_2 such that there is no connected component of infinite diameter in P_p.

  - Example: packing of shapes in $\mathbb{R}^n$ whose volume is proportional to the nth power of their diameter with a uniform proportionality constant.

- Theorem (P. 2020): There exists $p > 0$ such that the following holds. If G is a Benjamini-Schramm limit of (possibly random) finite planar graphs then there is no infinite cluster in G_p.

  - Used in study of the loop $O(n)$ model (Crawford–Glazman–Harel–P. 2020).
  - Main lemma: Benjamini–Schramm limits have circle packing with at most one accumulation point (small extension of Benjamini–Schramm (2001)).

- Remove one-ended and triangulation assumptions from result on recurrent planar graphs (in progress. Replaces He-Schramm theorem with Gurel-Gurevich–Nachmias–Souto 2017).

Conjectures (general circle packings)

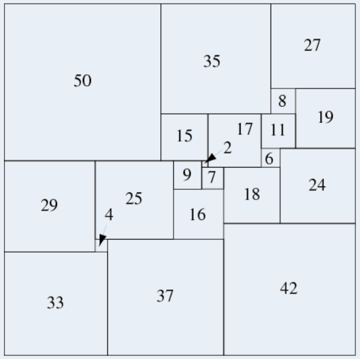
- Conjecture 1 ($p = 1/2$): No cluster of infinite diameter after $p = 1/2$ site percolation on any circle packing.
  Implies no infinite cluster after $\mathrm{p} = 1/2$ site percolation on recurrent one-ended triangulations (positive answer to Benjamini's question).

- Conjecture 2 (exponential decay):
  For each $p < 1/2$ there exists $f(p) > 0$ such that:
  Let P be a circle packing with $D \coloneqq \sup_{C \in P} \mathrm{diam}(C) < \infty$. Let $C_0 \in P$.
  After percolation with parameter p,
  $$\mathbb{P}(C_0 \text{ is in a cluster of diameter} \geq r) \leq \exp\left(-f(p)\frac{r}{D}\right).$$
  Implies existence of infinite cluster for $p > 1/2$ site percolation on transient bounded-degree one-ended triangulations (almost proves Benjamini's conjecture).

- Similar conjectures for ellipse packings (or other shapes).
  In conjecture 2, $f(p)$ is then replaced by $f(p, M)$ with M the maximal aspect ratio.
  Interesting to understand dependence on M, even for small p (has applications to the loop O(n) model).

Conjectures (critical percolation on circle packings)

- Let P be a circle packing representing a triangulation with carrier $\mathbb{R}^2$.
  If $D \coloneqq \sup_{C \in P} \operatorname{diam}(C) < \infty$, previous conjectures imply $p_c = 1/2$ (using duality).
  In fact, $p_c = 1/2$ may even hold under the assumption that the radii grow sublinearly (with a loglog correction) in the distance to the origin.
- For such circle packings, is the scaling limit of $p = 1/2$ site percolation the conformal loop ensemble CLE (as for the triangular lattice)?
- A related statement is to prove Russo–Seymour–Welsh type estimates at $p = 1/2$: the probability of a left-right crossing of a large rectangle by retained disks is in [c,1-c] where $c > 0$ depends only on the aspect ratio of the rectangle.
- Benjamini (2018) states a related conjecture: There exists $c > 0$ so that the following holds. Tile a square with squares of varying sizes so that at most three squares meet at corners. In $p = 1/2$ site percolation on the squares, the probability of a left-right crossing of retained squares is at least c.
- The presented results imply this when $p = 1/2$ is replaced by a universal constant sufficiently close to 1.

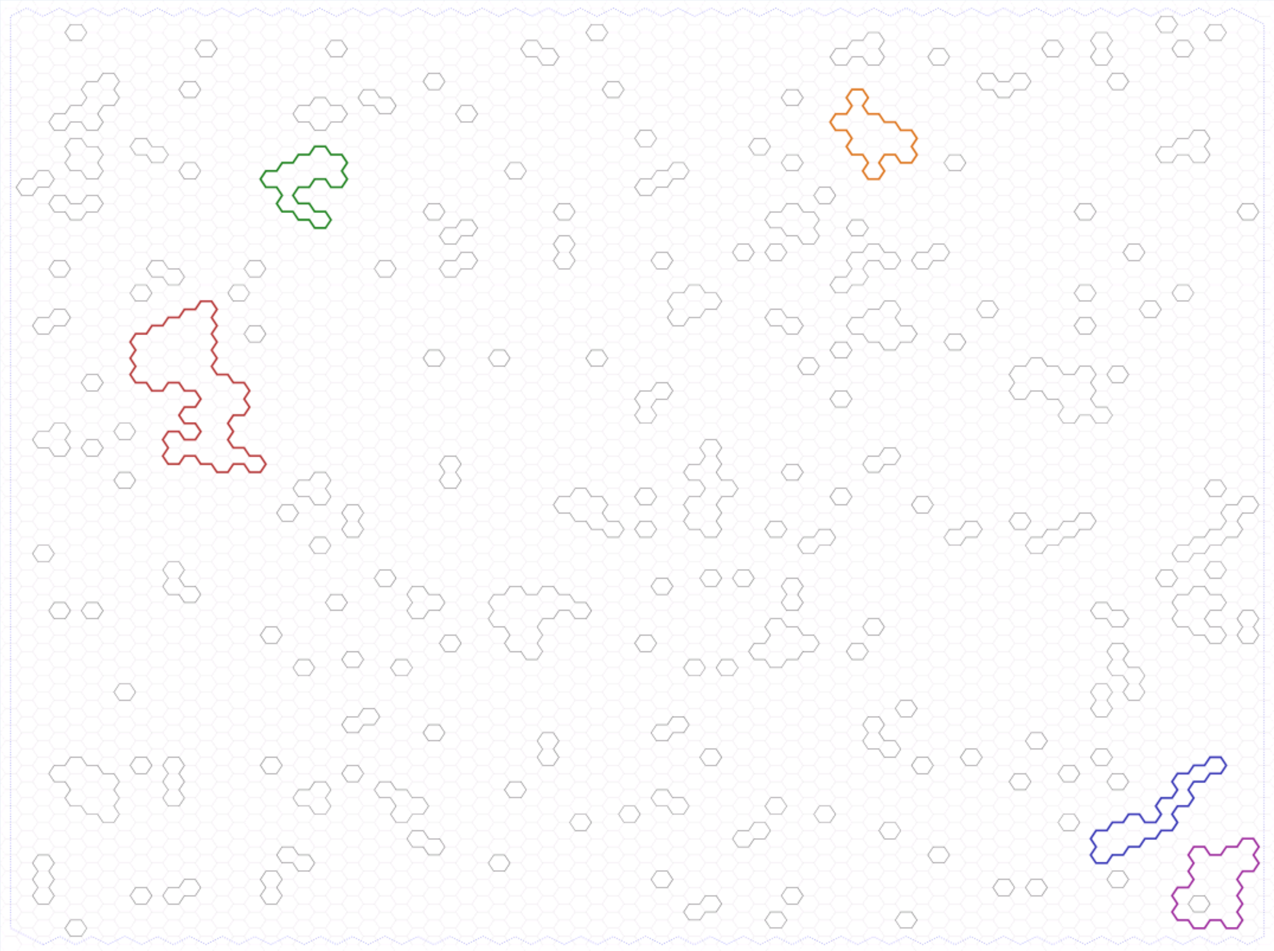


21-square perfect square

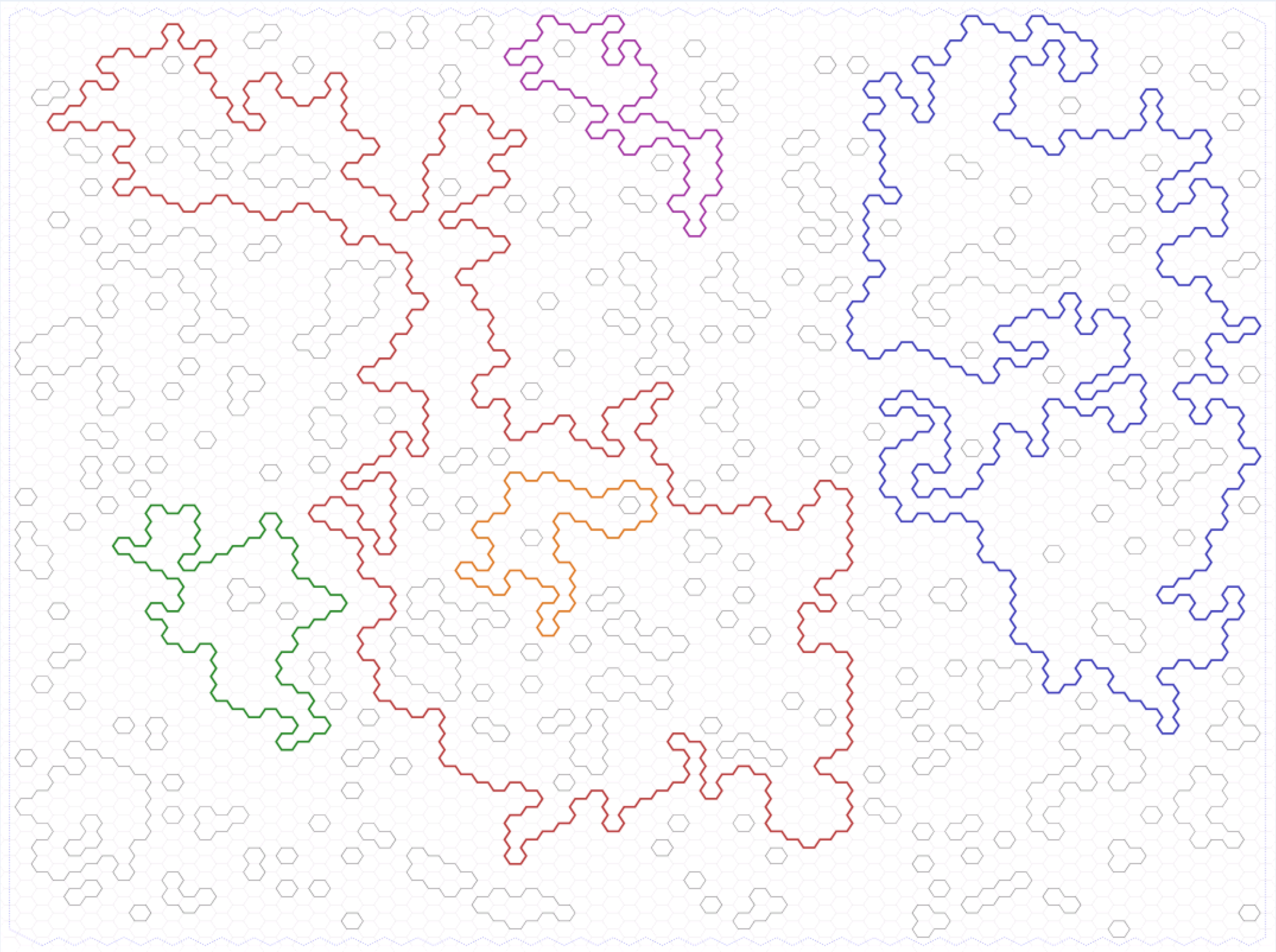
Loop O(n) model

- Model for non-intersecting loops on hexagonal lattice.
  Introduced by Domani-Mukamel-Nienhuis-Schwimmer (81) as an approximate graphical representation of spin $O(n)$ model.
  Gives a random-cluster (Fortuin-Kasteleyn) like representation for dilute Potts model with $q = n^2$ (Nienhuis 91).
- For given parameters $n, x > 0$, the weight of a configuration ω is given by $n^{L(\omega)}x^{|\omega|}$ where $L(\omega)$ is the number of loops in ω and $|\omega|$ is the number of edges in ω.
- Nienhuis (82) obtained the phase diagram of the model by mapping it to a Coulomb gas. Predicts critical behavior for $n \in [-2,2]$.

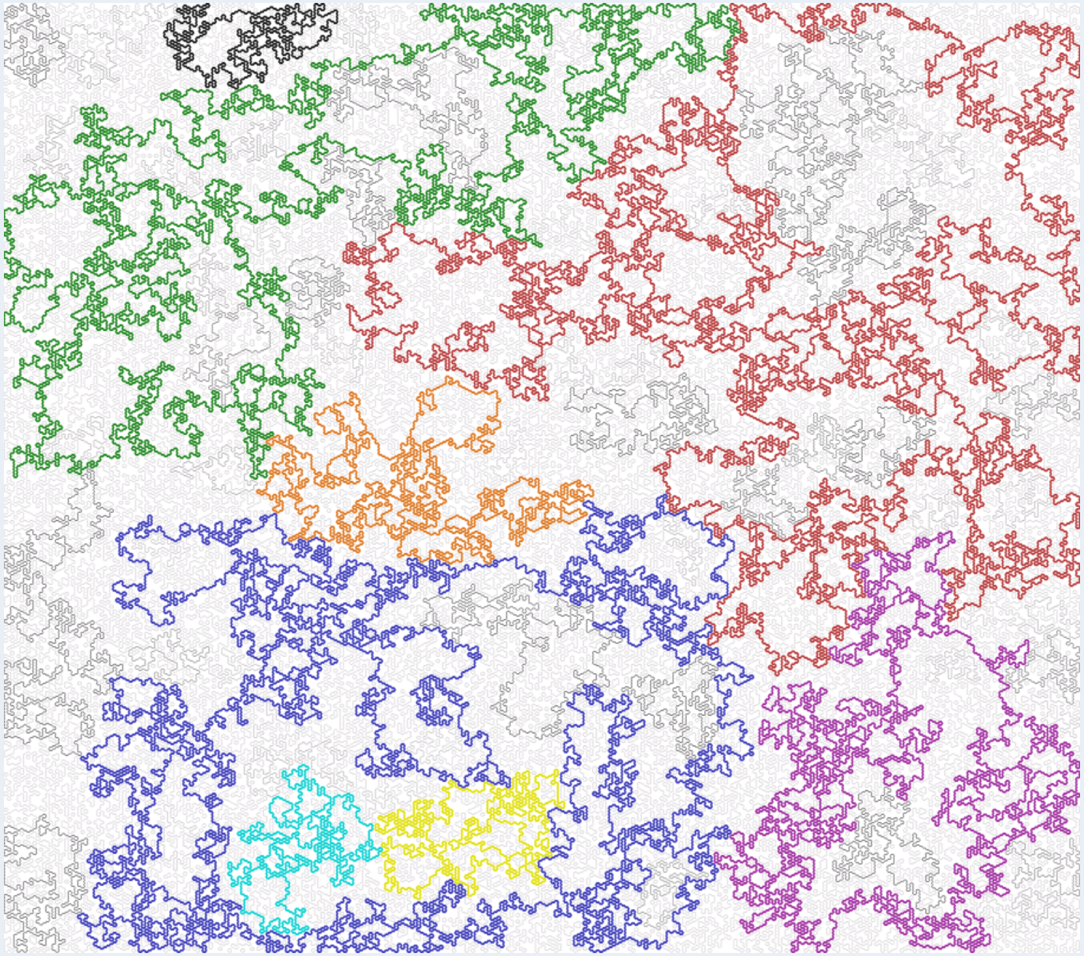
Loop $O(n)$: $n = 1.4, x = 0.57$



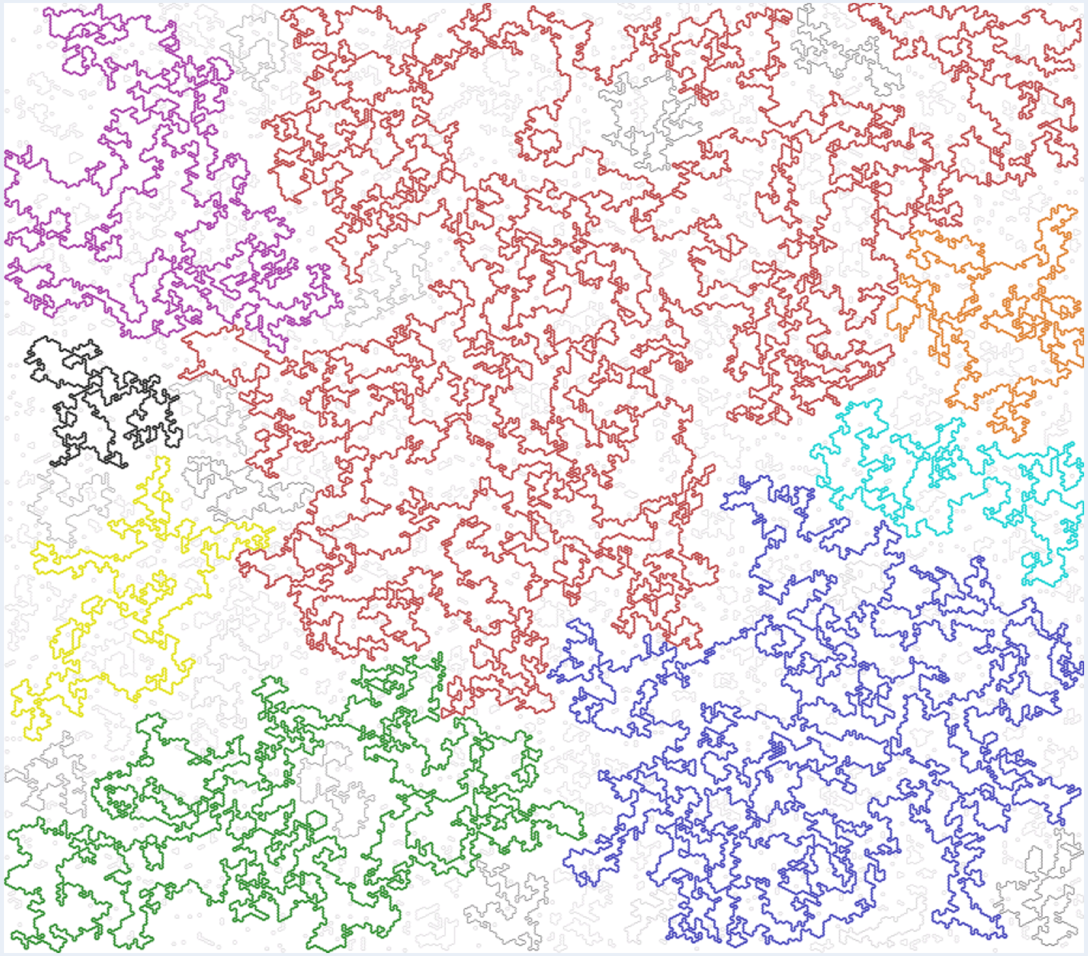
Loop $O(n)$: $n = 1.4, x = 0.63$



Loop $O(n)$: $n = 1.5, x = 1$

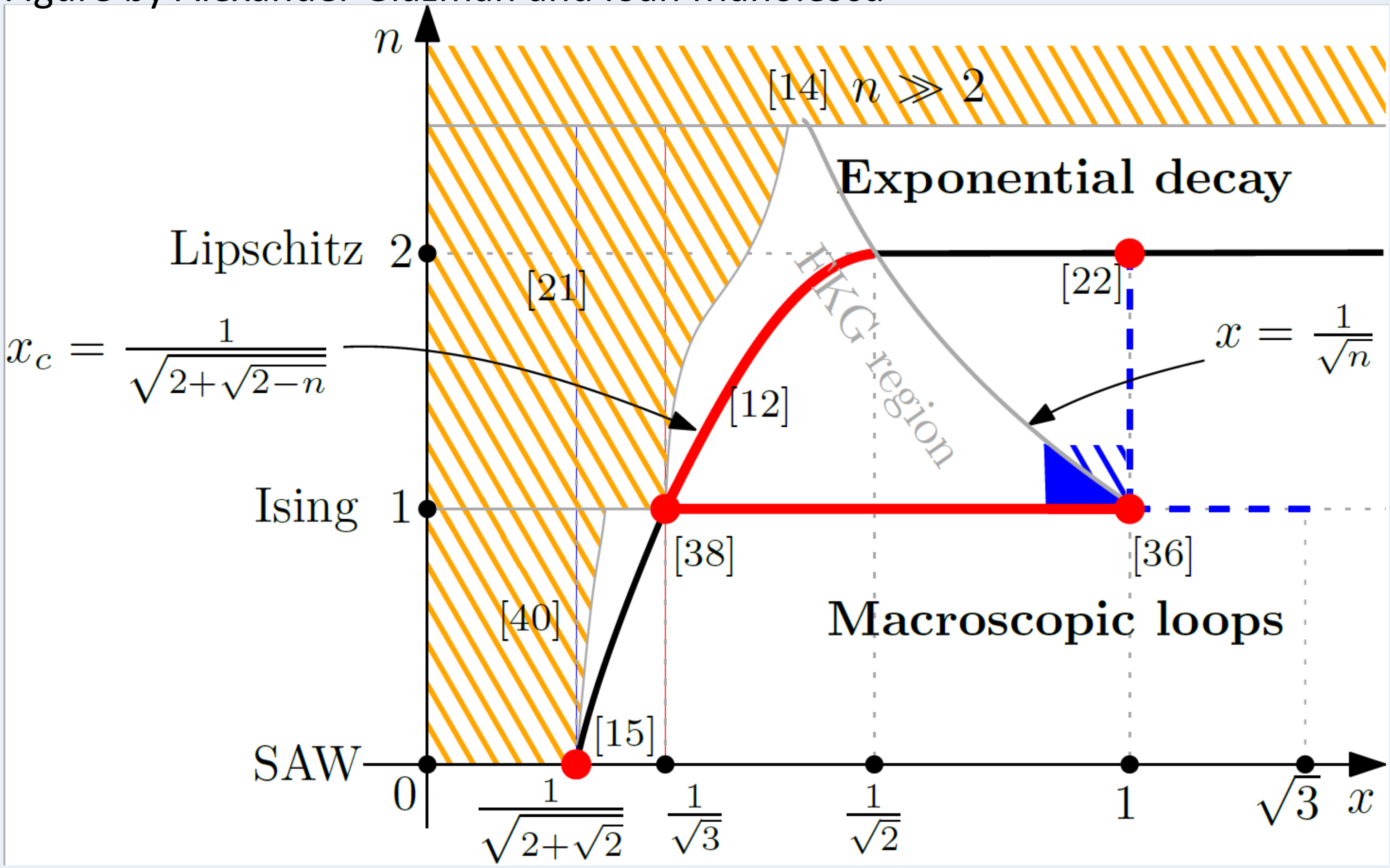


Loop $O(n)$: $n = 0.5, x = 0.6$



Predicted phase diagram and rigorous results

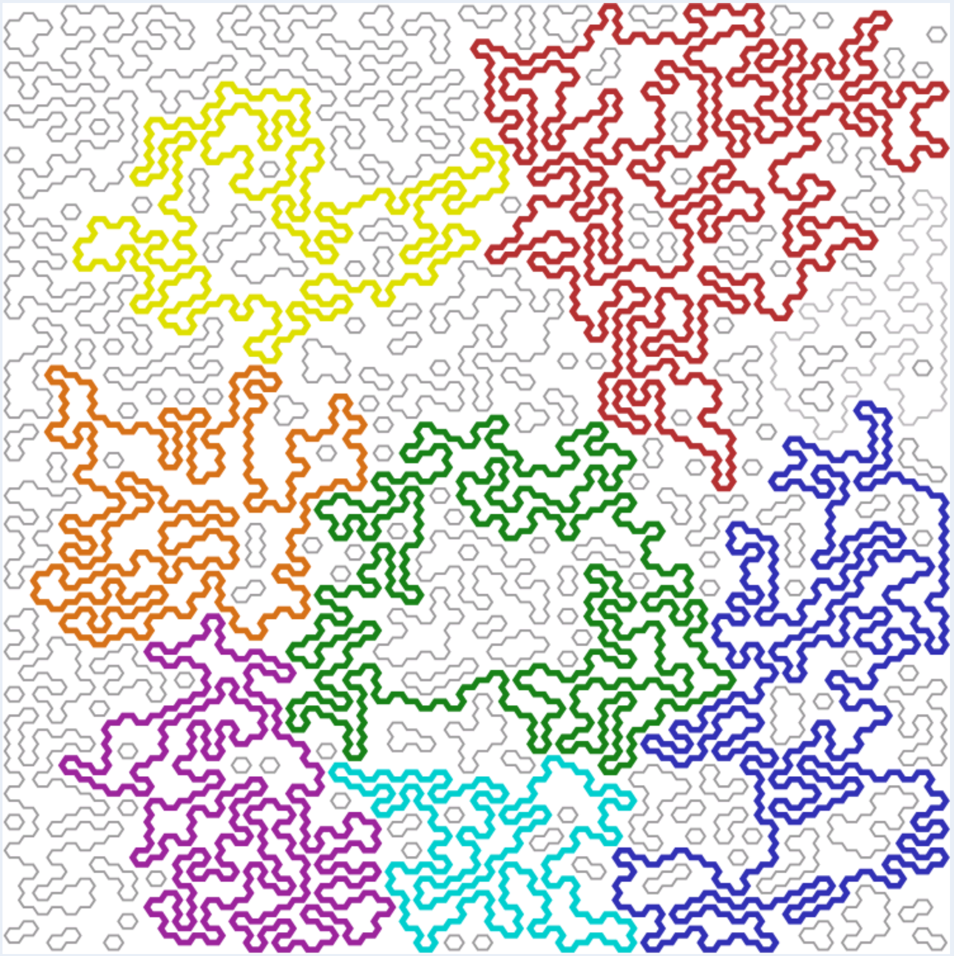
Figure by Alexander Glazman and Ioan Manolescu



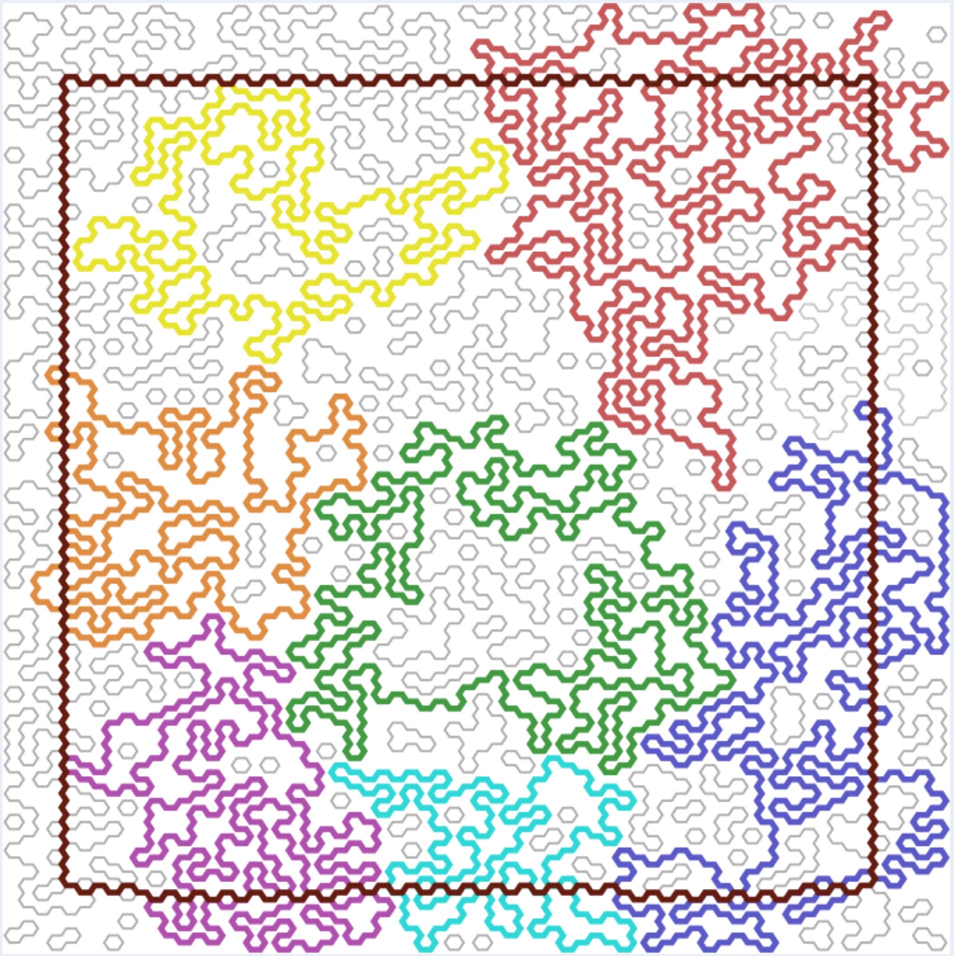
Crawford-Glazman-Harel-P. 2020: large loops in blue region of parameters.

Proof using XOR trick and result on no-percolation on circle packings.

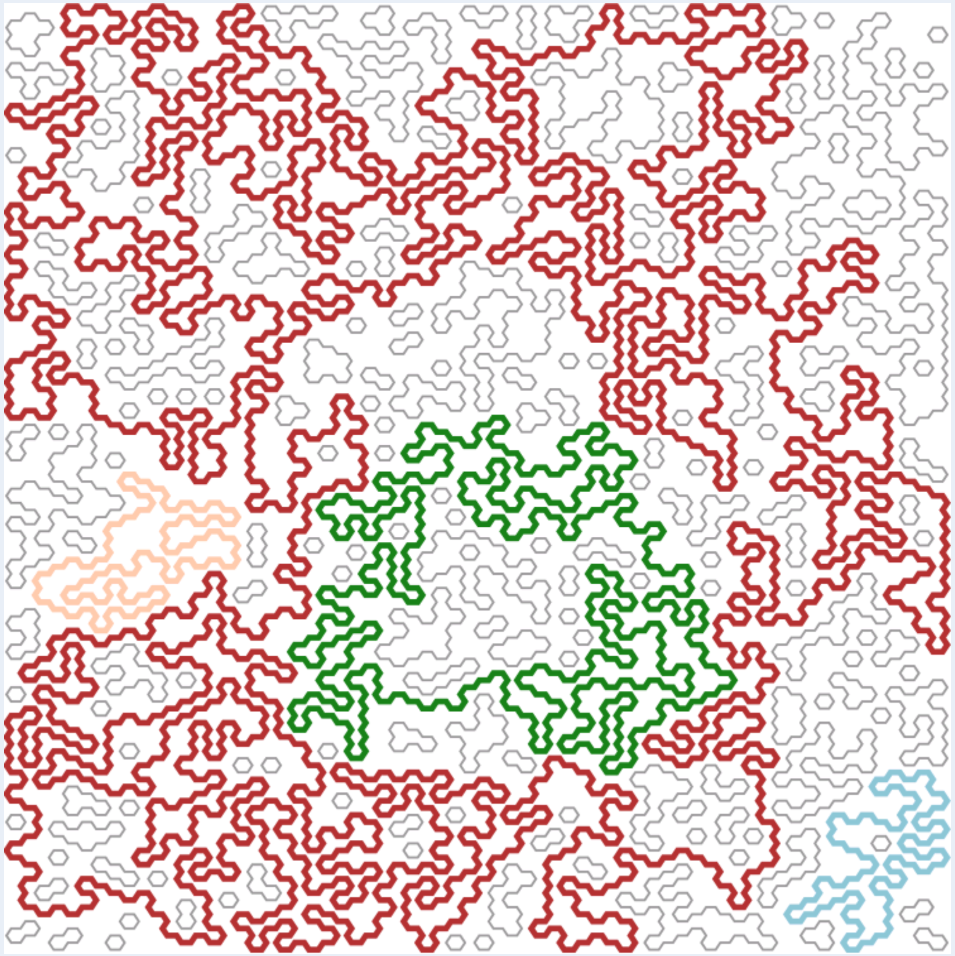
$n = x = 1$: critical percolation



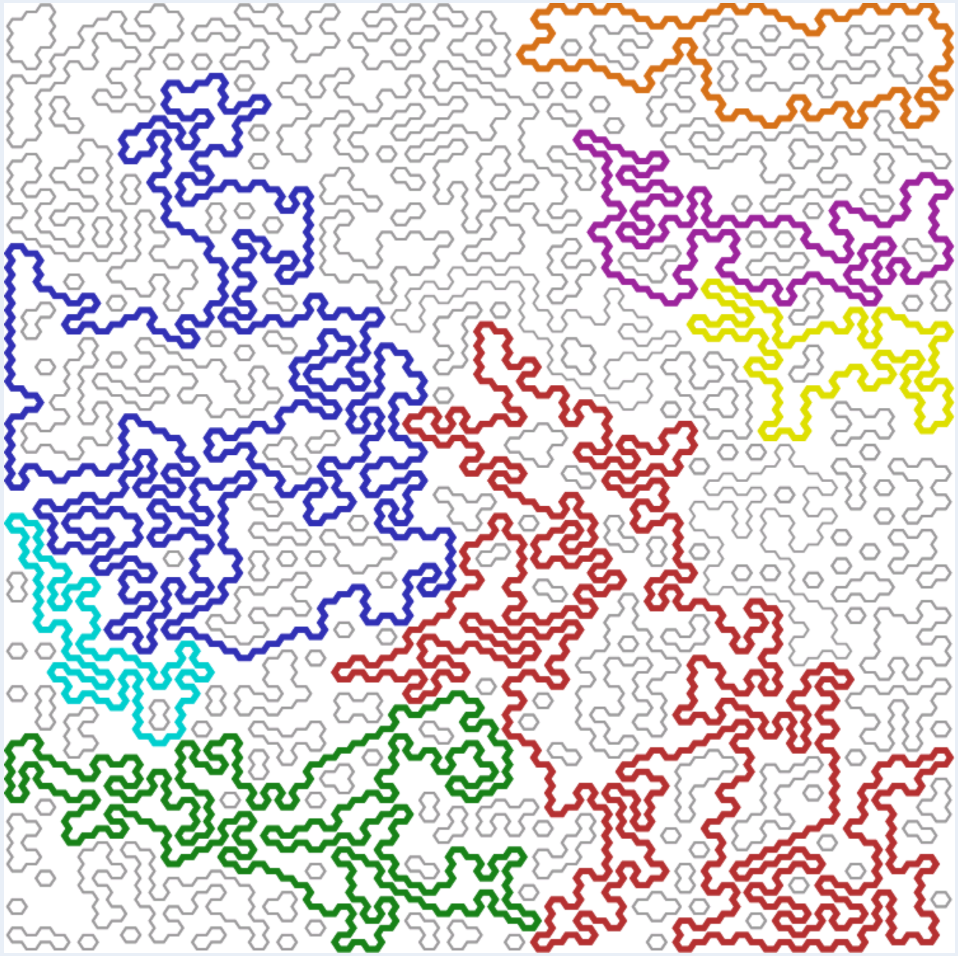
$n = x = 1$: XOR trick (1)



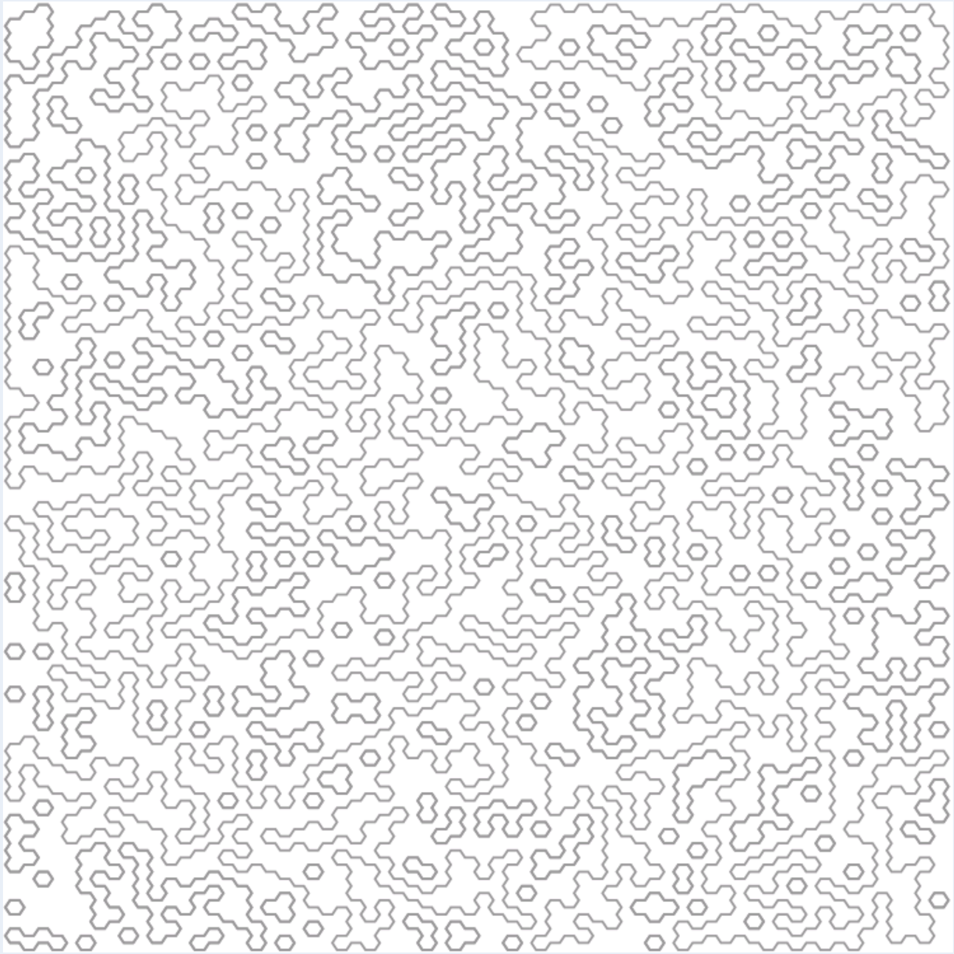
$n = x = 1$: XOR trick (2)



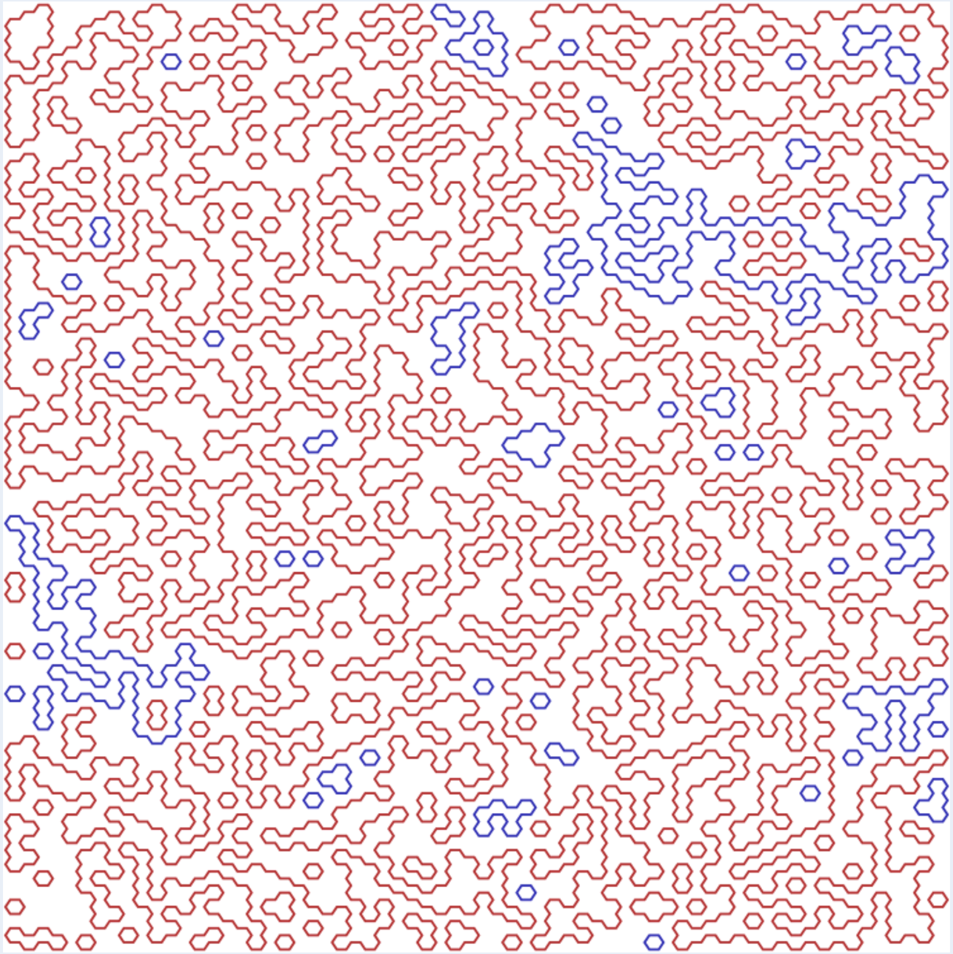
$n = 1.3, x = 1$:long loops colored



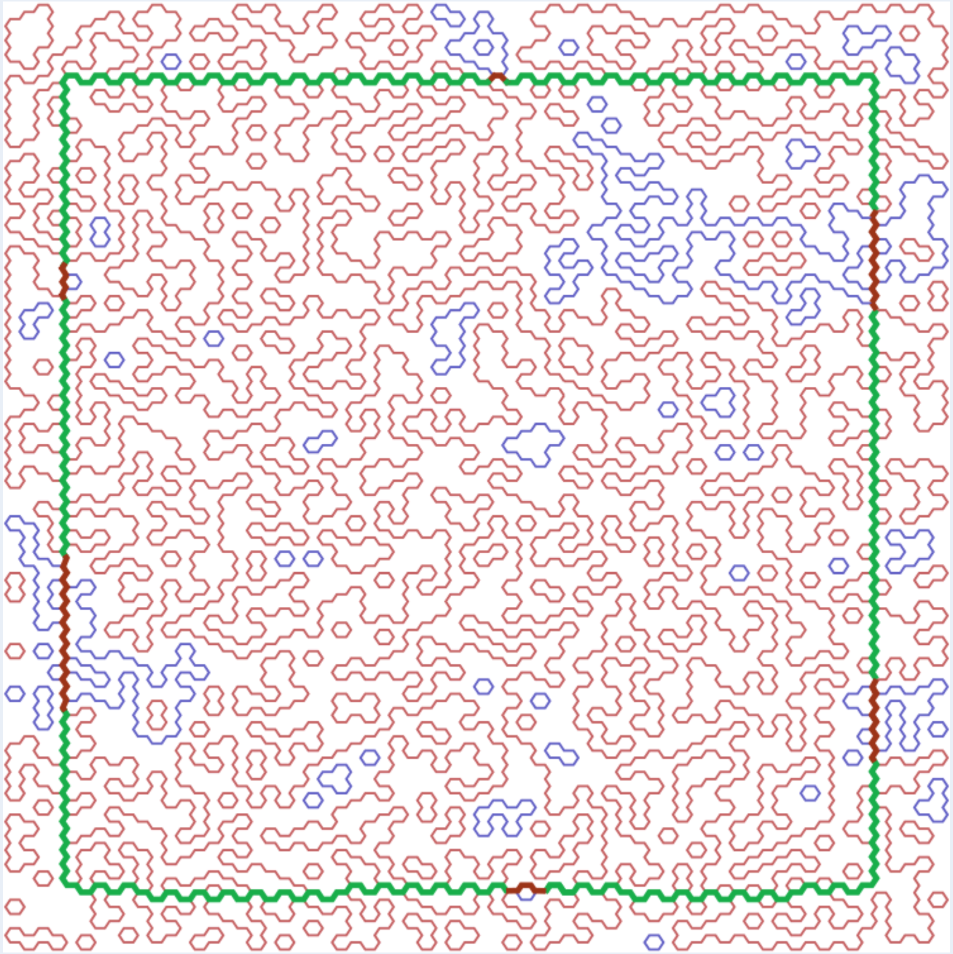
$n = 1.3, x = 1$: no colors



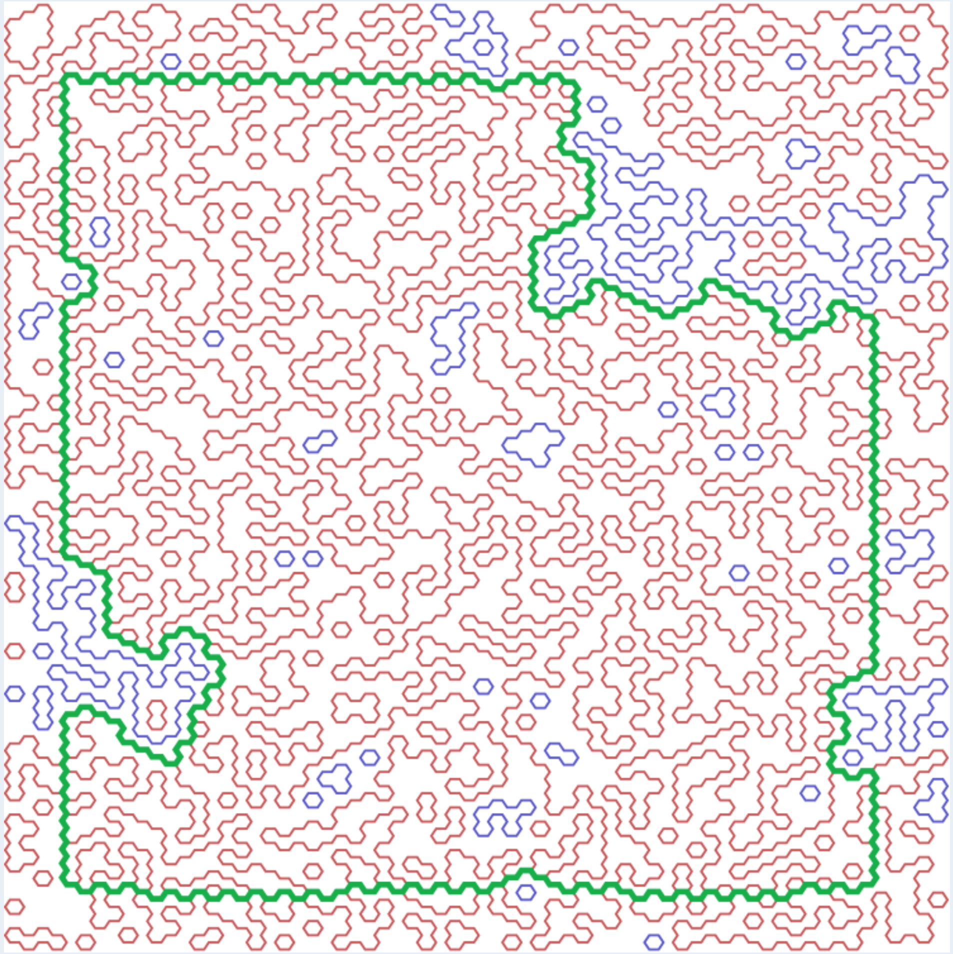
$n = 1.3, x = 1$: $\mathrm{p} = \frac{0.3}{1.3}$ loop percolation



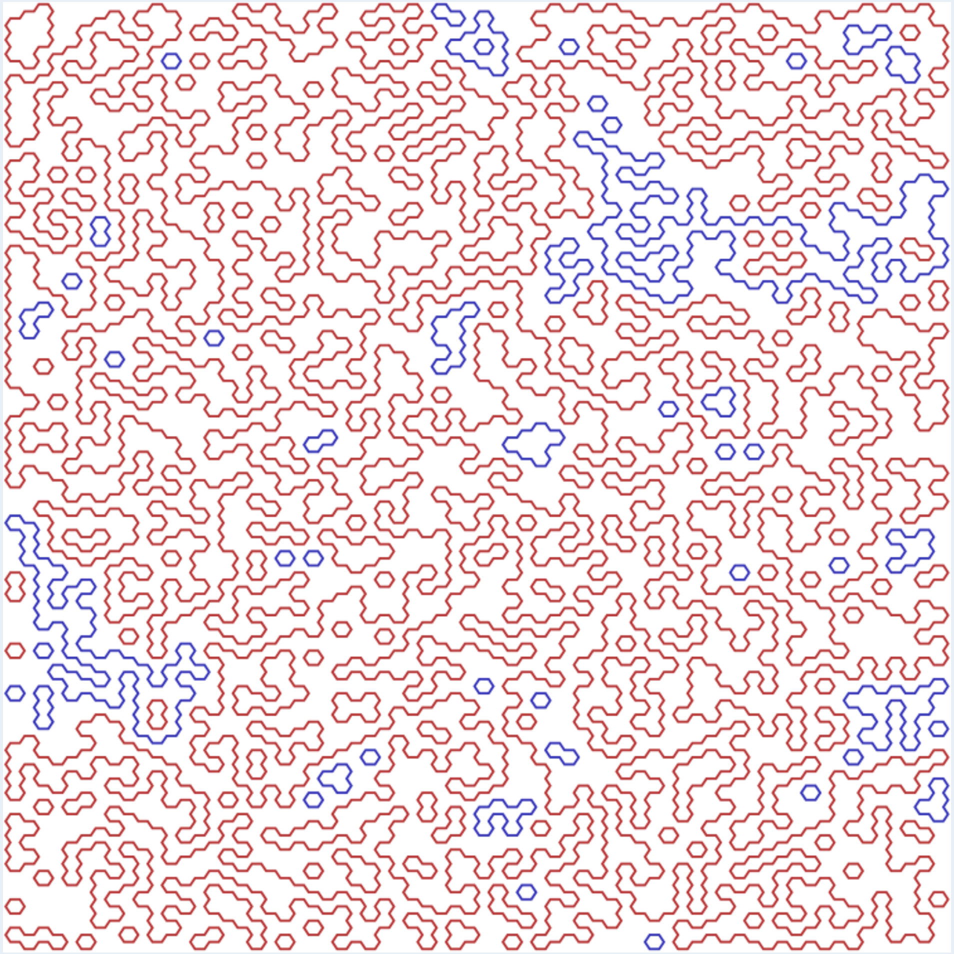
$n = 1.3, x = 1$: superimposed loop



$n = 1.3, x = 1$: adjusted loop



$n = 1.3, x = 1$: after XOR



$n = 1.3, x = 1$: final configuration